

Considerations on several approximate methods for the evaluation of adequate attenuation systems

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Abstract – Using a system with two degrees of freedom and one with three degrees of freedom it is determined the response of the systems using the β method of Newmark, the method of the central differences, methods of approximating the solution, by polynoms of the third degree and a method iteratively residual. The results are compared on the basis of the relative deviation for each point and on the basis of the average of the irregularity.

Keywords-*n-dimensional vibratory sistems, The calculation methods approximately, design of the residuem*

I. INTRODUCTION

THE MATHEMATICAL FORMULA OF THE ADEQUATE ATTENUATION SYSTEMS

The mathematical formula is obtained from the forces that act:

$$Ma = F(t) - F_{am} - F_{el} \quad (1)$$

To the force applied to the system $F(t)$ it is oppose a force of depreciation proportional to the speed $F_{am} = Bv = Bu$ and an elastic force proportional to the movement of $F_{el} = Ru$

Thus the dynamics of the dimensional adequate attenuation systems of n - is expressed by a differential matrix equation

$$M \cdot \ddot{u}(t) + B \cdot \dot{u}(t) + R \cdot u(t) = F(t), \quad (2)$$

where $u(t)$ is the column vector of the vibration of

$$\text{the } u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}. \quad (3)$$

The elements which define the real systems are:

$$M = \begin{pmatrix} m_1 & 0 & 0 & \dots & 0 \\ 0 & m_2 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & m_n \end{pmatrix} \quad \text{- The matrix of the inertial type component,} \quad (4)$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \vdots & & & & \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix} \quad \text{- The matrix of the dissipative type component} \quad (5)$$

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \dots & r_{1n} \\ r_{21} & r_{22} & r_{23} & \dots & r_{2n} \\ \vdots & & & & \\ r_{n1} & r_{n2} & r_{n3} & \dots & r_{nn} \end{pmatrix} \quad \text{- The matrix of the rigidity type component} \quad (6)$$

The column vector of the external forces is

$$F(t) = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix}. \quad (7)$$

It is considered a system (at $t = 0$) under the conditions (original)

$$u_0 = \begin{pmatrix} u_1(0) \\ u_2(0) \\ \vdots \\ u_n(0) \end{pmatrix},$$

$$\dot{u}(0) = v_0 = \begin{pmatrix} v_1(0) \\ v_2(0) \\ \vdots \\ v_n(0) \end{pmatrix}. \quad (8)$$

The time interval in order to calculate the solution of the differential equation is $t \in [0, b]$.

Use the equidistant nodes m located at $h = \Delta t = \frac{b}{m}$ relative to one another.

The test system with two degrees of freedom

[2]

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$

$$R = \begin{pmatrix} 6 & -5 \\ -5 & 5 \end{pmatrix}, F(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t). \quad (9)$$

The initial conditions are:

$$u(0) = U_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dot{u}(0) = V_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (10)$$

Physical system with two degree of freedom described by the mathematical model is shown in fig 1:

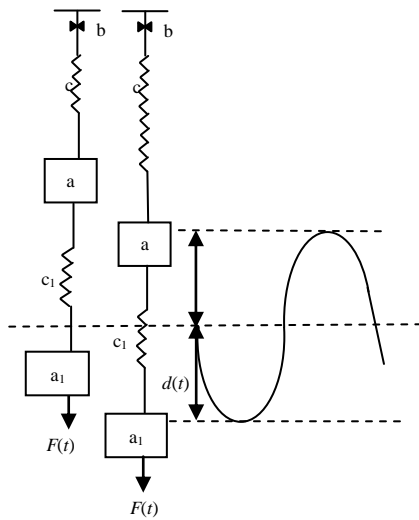


Fig. 1. Physical system with two degree of freedom

Using an analytical method for solving, for example the Laplace method [2] the vibration expression is obtained the according to the time function:

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \cos t \begin{pmatrix} -0,467 \\ -0,6 \end{pmatrix} + \sin t \begin{pmatrix} 0,4 \\ 0,467 \end{pmatrix} +$$

$$e^{-0,23169t} \cdot \left[\cos(0,6597t) \cdot \begin{pmatrix} 0,51957 \\ 0,5373 \end{pmatrix} + \sin(0,6597t) \cdot \begin{pmatrix} 0,973 \\ 1,128 \end{pmatrix} \right] +$$

$$e^{-1,267244t} \cdot \left[\cos(2,953855t) \cdot \begin{pmatrix} -0,05296 \\ 0,0626 \end{pmatrix} + \sin(2,953855t) \cdot \begin{pmatrix} 0,0036 \\ -0,0025 \end{pmatrix} \right] \quad (11)$$

The test system with three degrees of freedom [2]

$$M = \begin{pmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1,5m \end{pmatrix},$$

$$B = \begin{pmatrix} 16b & -4b & 0 \\ -4b & 8b & -2b \\ 0 & -2b & 11b \end{pmatrix},$$

$$R = \begin{pmatrix} 6k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 4k \end{pmatrix} \quad (12)$$

$$F(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos(t) \quad (13)$$

It is considered $m = 1000 \text{ kg}$, $b = 25 \text{ K}\cdot\text{N}\cdot\text{s}/\text{m}$ și $k = 1000 \text{ K}\cdot\text{N}/\text{m}$.

The initial conditions are:

$$u(0) = U_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \dot{u}(0) = V_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \quad (14)$$

II. THE PRESENTATION OF THE CALCULATION METHODS APPROXIMATELY TESTED

The calculation methods approximately tested are: β method of Newmark, method of the central differences, methods of approximating the solution by polynoms of the third degree, residual methods and approximation methods with sinusoidal functions.

A. β Method of Newmark

β method of Newmark [4] starts from the hypothesis that within the time investigated $\tau \in [t_i, t_{i+1}]$ the

acceleration is constant and equal to the average of the accelerations at the ends of the range width.

$$\ddot{u}(\tau) = \frac{1}{2}[\ddot{u}(t_i) + \ddot{u}(t_{i+1})] = \frac{1}{2}[\ddot{u}_i + \ddot{u}_{i+1}] \quad (15)$$

It is marked the range width with $h = t_{i+1} - t_i$. Relation for accelerations becomes

$$\ddot{u}(\tau) = \ddot{u}_i + \frac{\tau}{2}(\ddot{u}_{i+1} - \ddot{u}_i) \quad (16)$$

Using the original conditions ($\dot{u}(t_i) = \dot{u}_i$, $u(t_i) = u_i$) and integrating twice there are obtained the relations of recurrence which shall be modified by Newmark by inserting the coefficient β

$$\dot{u}_{i+1} = \dot{u}_i + \frac{h}{2}(\ddot{u}_i + \ddot{u}_{i+1}) \quad (17)$$

$$u_{i+1} = u_i + h\dot{u}_i + h\left(\frac{1}{2} - \beta\right)\ddot{u}_i + h^2\beta\ddot{u}_{i+1} \quad (18)$$

The standard form for the Newmark method is with $\beta = 1/4$.

In the work [1, 6] it shows that:
 ♣ for $\beta \geq 1/4$ method is stable no matter the chosen time frame;
 ♣ for $\beta < 1/4$, the method is stable only if the value of the report $c = h/T_n$ (T_n period of oscillation of the way their own order n) meets certain conditions as follows

if $\beta = 0$ must be that c would be $c < 0,318$,

if $\beta = 1/8$ must be that c would be $c \leq 0,450$, (19)

if $\beta = 1/6$ must be that c would be $c \leq 0,551$.

The application of the Newmark method in the case of adequate attenuation systems, described by the equation $M \cdot \ddot{u} + B \cdot \dot{u} + R \cdot u = F(t)$, in the initial conditions $u(0) = u_0$, $\dot{u}(0) = \dot{u}_0$, consists in the determination of the response at time t_1 (denoted by U_1) from the relationship

$$\begin{aligned} \left(M + \frac{h}{2} \cdot B + \beta \cdot h^2 \cdot R\right) \cdot U_1 = & \beta \cdot h^2 \cdot F_1 + M \cdot u_0 + h \cdot M \cdot v_0 + \\ & + h^2 \left[\left(\frac{1}{2} - \beta\right) \cdot I_n - h \cdot \left(\beta - \frac{1}{4}\right) \cdot B \cdot M^{-1} \right] \cdot (F_0 - B \cdot v_0 - R \cdot u_0) + \\ & + h^2 \cdot \left(\frac{1}{2} - \beta\right) \cdot B \cdot v_0 + \frac{h}{2} \cdot B \cdot u_0 \end{aligned} \quad (20)$$

The other components at the times t_2, \dots, t_n , shall be determined successively from the equation

$$\begin{aligned} \left(M + \frac{h}{2} \cdot B + \beta \cdot h^2 \cdot R\right) \cdot U_{i+1} = & h^2 [\beta \cdot F_{i+1} + (1 - 2 \cdot \beta) \cdot F_i + \beta \cdot F_i] + \\ & + [2 \cdot M - h^2 \cdot (1 - 2 \cdot \beta) \cdot R] \cdot U_i - \left[M + \frac{h}{2} \cdot B + \beta \cdot h^2 \cdot R\right] \cdot U_{i-1} \end{aligned} \quad (21)$$

For instance for t_2 ($i=1$) shall be calculated on the basis of $u_i = u_1$ (determined by the earlier relation), and depending on $u_{i-1} = u_0$ (known from the original conditions).

Observation. Relationships are array type, which means that the determination of the solution u_{i+1} involves reversing the matrix

$$A = \left(M + \frac{h}{2} \cdot B + \beta \cdot h^2 \cdot R\right) \quad (22)$$

B. The method of the central differences

The method of central differences [6] considers the vector $u(t)$ as a function of time for which it is written the development in Taylor series to the right and to the left of the point $t = t_i$

$$u(t_i + h) = u(t_i) + h \cdot \dot{u}(t_i) + \frac{h^2}{2} \cdot \ddot{u}(t_i) + \dots, \quad (23)$$

$$u(t_i - h) = u(t_i) - h \cdot \dot{u}(t_i) + \frac{h^2}{2} \cdot \ddot{u}(t_i) - \dots \quad (24)$$

Disregarding the superior terms from the Taylor development we find that the method of the central differences is the β method of Newmark for the $\beta = 0$. For the time t_1 response determination (denoted by U_1) is obtained from the relationship

$$\begin{aligned} \left(M + \frac{h}{2} \cdot B\right) \cdot U_1 = & M \cdot u_0 + h \cdot M \cdot v_0 + h^2 \cdot \left[\frac{1}{2} \cdot I_n + \frac{h}{4} \cdot B \cdot M^{-1}\right] \cdot \\ & \cdot (F_0 - B \cdot v_0 - R \cdot u_0) + \frac{h^2}{2} \cdot B \cdot v_0 + \frac{h}{2} \cdot B \cdot u_0 \end{aligned} \quad (25)$$

For the other times t_2, \dots, t_n , the answer is determined successively from the equation

$$\left(M + \frac{h}{2} \cdot B\right) \cdot U_{i+1} = h^2 \cdot F_i + (2 \cdot M - h^2 \cdot R) \cdot U_i + \left(\frac{h}{2} \cdot B - M\right) \cdot U_{i-1} \quad (26)$$

Observation. The method of the central differences as a particular case of β method of Newmark for $\beta = 0$, shows that it is necessary for maintaining stability, to have fulfilled the condition $c = h/T_n < 0,318$ [6] (where T_n is the period of yaw in its own way of order n).

C. Method A3A4

Time field $t \in [0, b]$ can be broken down into m

intervals of time $h = \Delta t = \frac{b}{m}$.

The method proposes the approximation of the vector on the go with a polynomial degree of the third degree for any interval of time $[t_i, t_{i+1}]$, $\forall i = 0, \dots, m$

$$u(t) = A_1 + A_2 t + A_3 t^2 + A_4 t^3 \quad (27)$$

Starting from the time interval $[t_0, t_0 + h]$ we consider initial conditions

$$u(0) = u_0, \quad \dot{u}(0) = v_0 \quad (28)$$

we provide that the resulting polynomial degree interpolation to verify that point. Get two of the constants of the model for the approximation, so that the resulting polynomial degree cannot be written in the form in

$$u(t) = u_0 + v_0(t - t_0) + A_3(t - t_0)^2 + A_4((t - t_0)^3 + 3t_0(t - t_0)^2) \quad (29)$$

In order to determine the other two unknowns (A_3, A_4) we provide that $u(t)$ to verify the model

in the points $t_0 + h$ and $t_0 + \frac{h}{2}$, which returns to the system of equations

$$M \ddot{u}(t_0 + \frac{h}{2}) + B \dot{u}(t_0 + \frac{h}{2}) + R u(t_0 + \frac{h}{2}) = F(t_0 + \frac{h}{2}) \quad (30)$$

$$M \ddot{u}(t_0 + h) + B \dot{u}(t_0 + h) + R u(t_0 + h) = F(t_0 + h)$$

We get the two equations with two unknowns

$$a_3 A_3 + a_{34} A_4 = F(t_0 + \frac{h}{2}) - B v_0 - R(u_0 + v_0 \frac{h}{2}) \quad (31)$$

$$a_{43} A_3 + a_4 A_4 = F(t_0 + h) - B v_0 - R(u_0 + v_0 h)$$

Where they knew each other

$$\begin{aligned} a_3 &= 2M + 2B \frac{h}{2} + R(\frac{h}{2})^2, \\ a_{34} &= 6M(t_0 + \frac{h}{2}) + 3B \frac{h}{2}(2t_0 + \frac{h}{2}) + R(\frac{h}{2})^2(3t_0 + \frac{h}{2}), \\ a_{43} &= 2M + 2Bh + R(h)^2, \\ a_4 &= 6M(t_0 + h) + 3Bh(2t_0 + h) + R(h)^2(3t_0 + h). \end{aligned} \quad (32)$$

Using the constants A_3, A_4 shall be calculated the movement and speed at $t = t_0 + h$, with the relations

$$u_\tau = u(t_0 + h) = u_0 + v_0 h + A_3 h^2 + A_4 h^2(3t_0 + h), \quad (33)$$

$$v_\tau = \dot{u}(t_0 + h) = v_0 + 2A_3 h + 3A_4 h(2t_0 + h). \quad (34)$$

D. Method A3A4 the smallest squares

Method A3A4 cmmmp is similar to the method A3A4 except for the fact that the method of the smallest squares to determine the constants of the A_3, A_4 .

The relations 27 -29 are valid in the case of this method.

The constants A_3, A_4 are determined by minimizing the objective function

$$F(A_3, A_4) = \sum_{k=1}^{m_2} (M \ddot{u}_k + B \dot{u}_k + R u_k - F_k)^2 = \min \quad (35)$$

It is used a division of the range $[t_0, t_0 + h]$ of m_2 spaced points relative to one another with the

$h_1 = \frac{h}{m_2}$, so that the points on which they are

calculated amount of relationship (35) are:

$$t_0, t_0 + h_1, \dots, t_k = t_0 + k h_1, \dots, t_0 + m_2 h_1 = t_0 + h \quad (36)$$

The method of the smallest squares leads to a system of two equations with two unknowns from which are derived constants A_3, A_4 (valid for range) A_3, A_4 .

For the calculation of the movement and speed are used relations (33),(34).

E. The iteratively - residual method

The time field $t \in [0, b]$ can be broken down into m equal time intervals or not. For the presentation of the proposed method we

consider the equal time intervals $h = \Delta t = \frac{b}{n}$.

The method proposes the approximation of the vector on the go with a polynomial degree of the third degree for any interval of time $[t_i, t_{i+1}]$, $\forall i = 0, \dots, n$, see relation (27).

Be imposed two conditions, namely polynomial to verify the initial conditions (28) to $t = t_0$ and

$$t = t_0 + \Delta t = t_0 + h$$

We give the values at the end of the range to

$$u(t_0 + \Delta t) = u_\tau, \quad \dot{u}(t_0 + \Delta t) = v_\tau \quad (37)$$

Obtain a resulting polynomial degree interpolation [5]

$$u(t_0 + \Delta t) = u_\tau, \quad \dot{u}(t_0 + \Delta t) = v_\tau \quad (38)$$

within which the interpolation expressions are:

$$\Phi_1(t) = 1 - \frac{3t^2}{(\Delta t)^2} + \frac{2t^3}{(\Delta t)^3}; \quad \Phi_2(t) = t - \frac{2t^2}{\Delta t} + \frac{t^3}{(\Delta t)^2}; \quad (39)$$

$$\Phi_3(t) = \frac{3t^2}{(\Delta t)^2} - \frac{2t^3}{(\Delta t)^3}; \quad \Phi_4(t) = -\frac{t^2}{\Delta t} + \frac{t^3}{(\Delta t)^2}. \quad (40)$$

The unknown constants A_3, A_4 have been replaced,

within the framework of the iteratively residual method -with two other unknown namely with the movement u_τ and speed v_τ .

For the purpose of determining u_τ , v_τ method uses Galerkin conditions [3], which require that the residue to be orthogonal to the functions of the interpolation

$$\begin{cases} \int_0^{\Delta t} \Phi_3(t) \cdot \varepsilon_i(t) dt = 0 \\ \int_0^{\Delta t} \Phi_4(t) \cdot \varepsilon_i(t) dt = 0 \end{cases} \quad (41)$$

where $\varepsilon_i(t)$ is the residue.

Note1. Using the model of the system and the function of approximation, from the relation (38), we obtain the calculated value

$$F_c(t) = M \cdot \ddot{u}(t) + B \cdot \dot{u}(t) + R \cdot u(t) .$$

The residue is the expression

$$\varepsilon_i(t) = F_c(t) - F(t) = \varphi_1 \cdot u_0 + \varphi_2 \cdot v_0 + \varphi_3 \cdot u_\tau + \varphi_4 \cdot v_\tau - F(t) \quad (42)$$

The conditions (41) determine a linear system from which there are derived the two unknowns (travel u_τ and speed v_τ).

On the following interval of time u_τ and v_τ represents the initial conditions, this allows you to resume calculations.

Note2. For integration there is not necessary to use an approximate method because the integrals contain some products of two algebraic functions $\Phi_3(t)$, $\Phi_4(t)$ are functions of the 3rd degree and can be expressed by

$$\Phi(\alpha_1, \alpha_2, t) = \alpha_1 t^2 + \alpha_2 t^3 \quad (43)$$

The second algebraic function is the

$$\Psi(\beta_1, \beta_2, \beta_3, \beta_4, t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^3 \quad (44)$$

The integral can be solved analytically, obtaining a polynomial degree

$$\int_{t_0}^{t_0+\Delta t} \Phi \Psi dt = \frac{\alpha_1 \beta_1}{3} (\Delta t)^3 + \frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{4} (\Delta t)^4 + \frac{\alpha_1 \beta_3 + \alpha_2 \beta_2}{5} (\Delta t)^5 + \frac{\alpha_1 \beta_4 + \alpha_2 \beta_3}{6} (\Delta t)^6 + \frac{\alpha_2 \beta_4}{7} (\Delta t)^7 \quad (45)$$

III. PRESENTATION OF THE RESULTS

The solution shall be made on a interval 25 seconds, with a $h=0.1s$, which means that are displayed 250 points.

The relative standard deviation shall be determined by the relationship

$$\varepsilon_k = \frac{u_{calculat} - u_{analitic}}{u_{analitic}} 100 \quad (46)$$

and the average value of the irregularity shall be calculated with

$$\varepsilon_{med} = \frac{1}{n} \sum_{k=1}^n abs(\varepsilon_k) \quad (47)$$

The test system with two degrees of freedom

In figure 2 there are shown the results for the system with two degrees of freedom. The analytical solution (11) is called "Analytical".

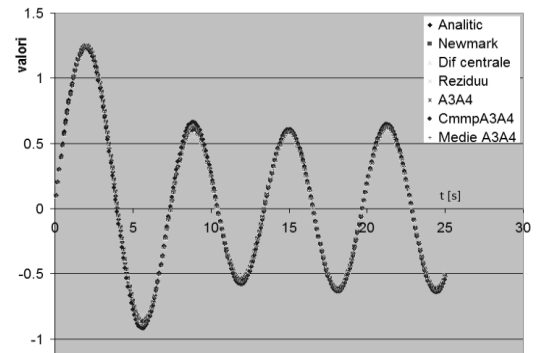


Fig. 2 Moving the system 1 according to the time

In figure 3 there are shown the percentage deviations of the value of the points approximated reported to the value determined analytically - with the equation (11), only for the methods A3A4 and A3A4cmmmp

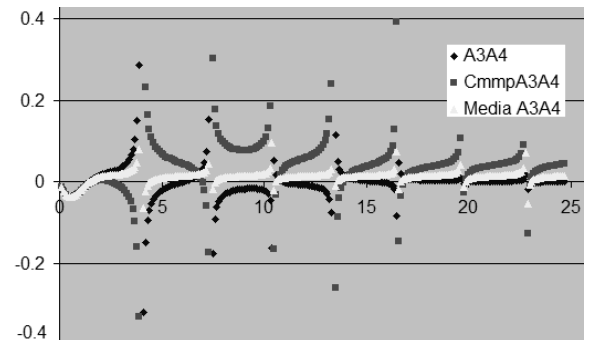


Fig. 3 The relative standard deviation in percentage for the A3A4

It is found that the irregularities of the two methods there are different when they pass through the zero of the analytical solution.

For the purpose of decreasing the irregularity has been used a weighted average mediation, giving a degree of trust more than the solution as determined by means of the A3A4 method, so that the curve called "Media A3A4" is calculated with the

$$\text{MediaA3A4} = (2 * A3A4 + C\text{mmpA3A4}) / 3 \quad (48)$$

In table 1 there are presented the average deviations in percentage for the 250 points.

TABLE 1

Method	The average deviation	Deviation at t=4s	Deviation at t=7.4s	Deviation at t=19.7s
Newmark	6.66	3.9	6.59	0.58
Central dif.	6.13	3.7	5.86	0.35
Reziduu	18.78	2.98	0.43	13.03
A3A4	6.11	3.73	5.81	0.23
A3A4cmmp	12.45	4.76	8.35	2.03
Med A3A4	2.81	0.9	1.09	0.82

In columns 3 to 5 there are presented the irregularities at the three values of the time where values are obtained large (usually at the passing through the zero of the analytical solution). It is found that the Newmark methods, differences in central and A3A4 lead to deviations comparable average at around 6 %.

The technique of averaging the weighted average of the results of the A3A4 and A3A4cmmp determines both decrease (twice) droop as environments and reducing the significant deviation when they pass through the zero of the solution.

The test system with three degrees of freedom

Figure 4 shows the calculation results for the test system with 3 degrees of freedom.

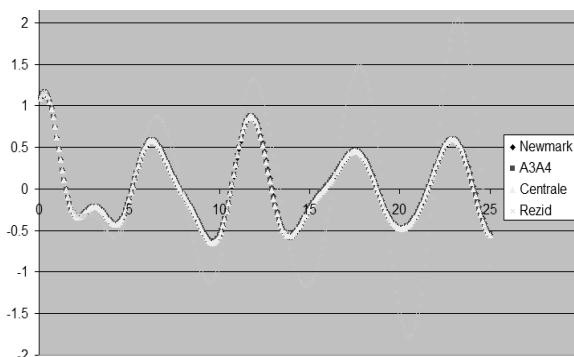


Fig. 4. Results for the system with 3 degrees of freedom

It is found that 3 of the methods lead to comparable results, and the residuum method lead to oscillations are maintained.

It can be concluded that the residue method cannot be applied in the case of systems with more than 2 degrees of freedom.

Calculating the relative deviations of a method of each other get Newmark-Diferencial you 8%, Newmark-A3A4 11.73%, central differences-A3A4 3.97%. We can reach the conclusion that the method of the central differences and the method A3A4 lead to results (within the limit of 4%).

IV. CONCLUSIONS

Using the test with two and three degrees of investigations have been obtained comparable results for the Newmark methods, central differences and a new method of A3A4 proposed in the framework of the work.

The method of design of the residue on the functions of the interpolation (Galerkin) leads to large deviations in the case of the test with two degrees of freedom (17.78%) and does not lead to a fair solution to the test with three degrees of freedom.

REFERENCES

- [1]Connor J.J., Brebia C.A., *The finite Element Technique*, Santhamton, 1976;
- [2]Dinu O. *A system of automatic acquisition of data in real time for monitoring the vibration*, UNITG.-PhD paper, manuscript, 2015;
- [3]Dorn W.S., Mc Cracken D.D., *Numerical Methods with programs in FORTRAN IV*, „Tehnica” Publisher House, București, 1976.
- [4]Newmark, N. M., *A method of computation for structural dynamics*, Journal of Engineering Mechanics, ASCE, 85 (EM3) 67-94, 1959.
- [5]Posea N., *The dynamic calculation of the structures*, “Tehnica” Publishing House, Bucharest, 1981;
- [6]Warburton G. B., *The Dynamical behavior of structures*, Pergamum Press, London, 2014;